L(2,1)-labeling of Complete Multipartite Graphs

***Yuri C Sagala*[1]*, Susiana*[2]**

[1,2]Department of Mathematics,

State University of Medan,

Medan, Indonesia

ABSTRACT: The λ-number of a graphs *G* is the minimum value λ such that *G* admits a labeling with labels from {0,1,..., λ} where vertices at distance two get different labels and adjacent vertices get labels that are at least two apart. Complete multipartite graph (or complete *p*-partite graph) $K_{a_1,a_2,\dots,a_p}$ is that graph *G* whose vertex set can be partitioned into *p* subsets $A_1, A_2,\dots, A_p$ (also called partite sets) with $|A_i| = a_i$ for $1 \le i \le p$, and $|A_i|$ is the cardinality of each partite sets, such that uv ϵ E(G) if u ϵ A_i and v ϵ A_j, where 1 ≤ i, j ≤ p, and i ≤ j. In this paper, we show that the smallest maximum labeling number of complete *p*-partite graphs is $\sum_{i=1}^{p} a_i + (p-2)$.

KEYWORDS: L(2,1)-labeling, complete multipartite graphs, λ-number

1. INTRODUCTION

An *L(2,1)*-labeling of a graph *G* is a function *f* from the vertices set *V(G)* to the set of all nonnegative integers such that if *dist(u,v) = 1* then $|f(u) - f(v)| \ge 2$, and if *dist(u,v) = 2* then $|f(u) - f(v)| \ge 1$, where *dist(u,v)* denotes the distance between *u* and *v* in *G*. The *L(2,1)*-labeling number *λ(G)* of *G* is the smallest number λ such that *G* has an *L(2,1)*-labeling with max {f(v) : v ϵ V(G)} = λ.

The *L(2,1)*-labeling concept arose from the problem of assigning frequencies of radio transmitters. To avoid interferences, transmitters that are close must receive frequencies that are sufficiently apart. Motivated by this problem, [8] proposed *L(2,1)*-labeling. Other problem was proposed by proposed by [9] is to minimize the span of frequencies.

Complete multipartite graphs were introduced by [2] and [3]. One of variation of complete multipartite graphs is bipartite graphs that introduced by [18], [15] and [14]. Bipartite graph is a graph whose vertex-set can be partitioned into two subsets of vertices. But, if the partition of its vertices is *p*, it is called as *p*-partite graph, for *p ≥ 2*.

In section 2 we introduce the multi-partite graphs of interest and describe preliminary lemmas. In the last section, we give the *L(2,1)*-labeling number of complete multi-partite graphs, or $\lambda(K_{a_1,a_2,a_3,\dots,a_p})$.

2. PRELIMINARIES

Definition 2.1 [2] *Complete multipartite graph (or complete p-partite graph)* $\boldsymbol{K_{a_1,a_2,a_3,\dots,a_p}}$ *is that graph G whose vertex set can be partitioned into p subsets* $A_1, A_2,\dots, A_p$ *(also called partite sets) with* $|A_i| = a_i$ *for* $1 \le i \le j$*, such that uv ϵ E(G) if u ϵ* A_i *and v ϵ* A_j*, where 1 ≤ i, j ≤ p, and i ≠ j.*

By other words, complete multipartite graphs has vertices set $A_1, A_2,\dots, A_p$. If *i ≠ j*, all vertices in A_i are adjacent to all vertices in A_j. But if *i = j*, there is none edge. So, A_i is null graph or complement of complete graph. If we sum A_i and A_j, we get a graph that isomorphic with complete multipartite graph, so complete multipartite graph can be also constructed by summing *p* null graphs, or $\boldsymbol{K_{a_1,a_2,a_3,\dots,a_p}} = A_1 + A_2 + \dots + A_p$. Cause of that, complete multipartite graph has $\sum_{i=1}^{p} a_i$ vertices and has $\sum_{i \neq j}^{p} a_i \cdot a_j$ edges. If p = 2, it is called as complete bipartite graphs. The complete bipartite graphs is shown in this fig. 1 below.

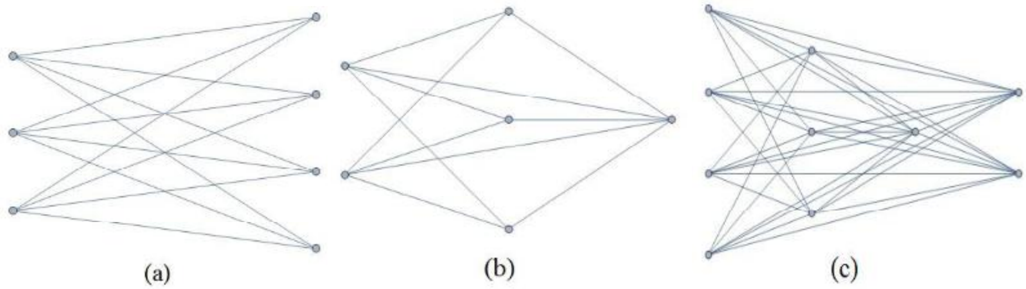

Fig. 1. ***(a) Complete bipartite graph $K_{3,4}$, (b) Complete tripartite graph $K_{2,3,1}$, and (c) Complete tetrapartite graph $K_{4,3,2,1}$***

From the fig. 1 (a), there are two partition of vertices A_1 and A_2, $a_1 = 3$ and $a_2 = 4$. We take $u \in A_1$ and $v \in A_2$. Vertices u_x and u_y are not adjacent, because they are in same set. But u and v are adjacent, because they are in different sets. By operating two graphs, A_1 is $\overline{K_3}$, A_2 is $\overline{K_4}$. By sum this graph, $\overline{K_3} + \overline{K_4}$, we get bipartite graph $K_{3,4}$. And the fig. 1 (b) is also, It is constructed from $\overline{K_2} + \overline{K_5}$.

So that, for more partite sets, $K_{a_1,a_2,a_3,\ldots,a_p} = \overline{K_{a_1}} + \overline{K_{a_2}} + \cdots + \overline{K_{a_p}}$.

Lemma 2.1 *Let $\chi(G)$ is a chromatic number of graph G, the chromatic number of complete p-partite graphs $\chi\left(K_{a_1,a_2,a_3,\ldots,a_p}\right) = p$.*

Proof. Because no adjacent vertices in one sets, we can give color them with same color. But we must give different color to vertices in another sets, because they are adjacent. So, the chromatic number of complete multipartite graphs is number of its partite sets, or in other words, the complete p-partite graphs can be partitioned in to *p*-colors. ■

Lemma 2.2 $K_{\underbrace{1,1,\ldots,1}_{p}}$ *is complete graph with p vertices.*

Proof. $K_{\underbrace{1,1,\ldots,1}_{p}}$ has *p* subsets of its vertices, and K_1 is self-complementary. By definition 2.1, every two distinct vertex of this graph is adjacent, because $a_i = 1$. So, $K_{\underbrace{1,1,\ldots,1}_{n}} = K_n$. ■

Lemma 2.3 $diam(K_{a_1,a_2,a_3,\ldots,a_p}) = \begin{cases} 1, & \forall\, a_i = 1 \\ 2, & otherwise \end{cases}$

Proof. By lemma 2.1, $diam(K_n) = 1$, because every two distinct vertex of complete graph is adjacent. But, if there exist $a_i \neq 1$, it means the farthest vertex of $u \in A_i$ is a vertex in A_i also, bbecause all vertices in A_i is adjacent to all vertices in A_j. If $u \in A_i$, u_x and u_y are not adjacent, caused by definition 2.1. Because u_x and u_y are separated by $v \in A_j$, $dist(u_x, u_y) = 2$. Because the farthest vertex of If $u \in A_i$ is also in A_i, $diam\left(K_{a_1,a_2,a_3,\ldots,a_p}\right) = 2$. ■

3. L(2,1)-LABELING ON COMPLETE MULTIPARTITE GRAPHS

Lemma 3.1 *No two vertices of complete multipartite graphs have a same label.*

Proof. L(2,1)-labeling, we only consider to give label of two adjacent vertices or in distance two, but if the distance is greater or equal than 3, no restriction is placed. Lemma 2.3 resulted labels of vertices in same set are differ at least by one, and label of vertices in different sets are differ at least by two. So, no two vertices have a same labels in complete graphs. ■

Theorem 3.1 *Let λ(G) is the smallest maximum L(2,1)-labelling number in graph G, the smallest maximum L(2,1)-labeling number of complete p-partite graph $\lambda\left(K_{a_1,a_2,\ldots,a_p}\right) = \sum_{i=1}^{p} a_i + (p-2)$.*

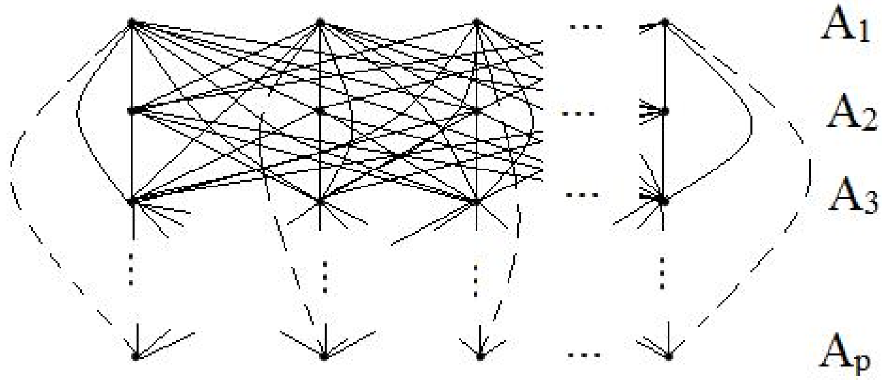

Fig. 2. Complete p-partite graph

Proof. Let the vertices in A_1 are labeled with *0, 1, 2, ..., a_1-1*. By lemma 3.1, no two vertices in A_1 given a same label, so a_1-1 is the smallest maximum label in A_1.

- For $p = 2$
  We start from $p = 2$ because it is the smallest partition of this graph. Because all vertices in A_2 are adjacent to all vertices in A_1, so the label in A_2 starts from $(a_1 - 1) + 2 = a_1 + 1$.
  And, the other vertices in A_2 can be labelled with $((a_1 +1) +1)$, $((a_1 + 1) + 2)$, ..., $((a_1 + 1) + (a_2 - 1))$ because the vertices in distance two can be given label differ by less or equal than one, and cannot given label which less or equal than $a_1 + a_2 + 1$ because the rule of L(2,1)-labeling. So, the smallest maximum labelling number $\lambda(K_{a_1,a_2,}) = a_1 + a_2$.
- For $p = k$
  Because all vertices in A_p are adjacent to all vertices in A_i, for $1 \le i \le k-1$, so the label in A_p starts from $(a_1 + a_2 + a_3 + ... + a_{k-1}) + (k-1)$.
  And, the other vertices in A_2 can be labelled with $(((a_1 + a_2 + a_3 + ... + a_{k-1}) + (k-1) + 1)$, $(((a_1 + a_2 + a_3 + ... + a_{k-1}) + (k-1) + 2)$, ..., $(((a_1 + a_2 + a_3 + ... + a_{k-1}) + (k-1) + (a_k - 1))$ because the vertices in distance two can be given label differ by one, and cannot given label which less or equal than $a_1 + a_2 + a_3 + ...+ a_{k-1} + a_k + (k-2)$ because the rule of L(2,1)-labeling, so the smallest maximum labelling number $\lambda(K_{a_1,a_2,...,a_k}) = \sum_{i=1}^{k} a_i + (k-2)$
- For $p = k+1$
  Because all vertices in A_{p+1} are adjacent to all vertices in A_i, for $1 \le i \le k$, so the label in A_p starts from $(a_1 + a_2 + a_3 + ... + a_k) + k$.
  And, the other vertices in A_2 can be labelled with $(((a_1 + a_2 + a_3 + ... + a_k) + k + 1)$, $(((a_1 + a_2 + a_3 + ... + a_k) + k + 2)$, ..., $(((a_1 + a_2 + a_3 + ... + a_k) + k + (a_{k+1} - 1))$ because the vertices in distance two can be given label differ by one, and cannot given label which less or equal than $a_1 + a_2 + a_3 + ...+ a_{k-1} + a_k + a_{k+1} + ((k+1) - 2)$ because the rule of L(2,1)-labeling, so the smallest maximum labelling number $\lambda\left(K_{a_1,a_2,...,a_p,a_{p+1}}\right) = \sum_{i=1}^{p+1} a_i + (p-1)$

  From these steps, we get a L(2,1)-labeling pattern. So, the smallest maximum labeling number of complete p-partite graph or $\lambda\left(K_{a_1a_2...a_p}\right) = a_1 + a_2 + \cdots + a_p + (p-2)$. For simplicity, $\lambda\left(K_{a_1a_2...a_p}\right) = \sum_{i=1}^{p} a_i + (p-2)$. ■

Corollary 3.1 [6], [11] $\lambda(K_p) = 2p - 2$.

Proof. By lemma 2.1, complete graph is one variation of complete p-partite graph with $a_i = 1$, and we get the smallest labeling number of complete multipartite graphs from theorem 3.1. So,

$$\lambda(K_p) = \underbrace{1 + 1 + \cdots + 1}_{p} + (p-2)$$
$$\lambda(K_p) = p + (p-2)$$
$$\lambda(K_p) = 2p - 2$$

So, the labeling number of complete graph with p vertices is $\mathbf{2p-2}$. ■

Corollary 3.2 [11] *For C_n is a cycle graph with n vertices, $\boldsymbol{\lambda(C_4)=4}$.*

Proof. C_4 is a complete bipartite graph $K_{2,2}$, so

$$\boldsymbol{\lambda(C_4) = 2+2+(2-2)}$$
$$\boldsymbol{\lambda(C_4) = 4+(2-2)}$$
$$\boldsymbol{\lambda(C_4) = 4}$$

So, the labeling number of C_4 is 4. ■

Corollary 3.3 [11] *For P_n is a path graph with n vertices, $\boldsymbol{\lambda(P_3)=3}$.*

Proof. P_3 is a complete bipartite graph $K_{1,2}$, so

$$\boldsymbol{\lambda(P_3) = 2+1+(2-2)}$$
$$\boldsymbol{\lambda(P_3) = 3+(2-2)}$$
$$\boldsymbol{\lambda(P_3) = 3}$$

So, the labeling number of P_3 is $\mathbf{3}$. ■

4. Conclusion

In this paper, we find the smallest maximum L(2,1)-labeling number of complete multipartite graphs with p-partitions is $\boldsymbol{\lambda\left(K_{a_1,a_2,\dots,a_p}\right) = \sum_{i=1}^{p} a_i + (p-2)}$.

References

[1] T. Calamoneri. and R. Petreschi, "L(2,1)-labeling of Unigraphs," *Department of Computer Science*, pp. 1–19, 2009.
[2] G. Chartrand and P. Zhang, *Chromatic Graph Theory*, CRC Press, 2009.
[3] N. Biggs, *Algebraic Graph Theory,* 2nd Ed., Cambridge University Press, 1993.
[4] G. J Chang and D. Kuo, "The L(2,1)-Labeling Problem on Graphs," *SIAM J. Disc. Math,* vol. 9, pp. 309–316, 1996.
[5] Chartrand G., and Lesniak L., *Graphs and Digraphs,* CRC Press, 1996.
[6] H. Y. Fu and D. Xie, "Equitable L(2,1)-labelings of Sierpiński Graphs," *Australasian Journal of Combinatorics,* vol. 46, pp. 147–156, 2010.
[7] S. Gravier, S. Klavžar, and M. Mollard, "Codes and L(2,1)-labelings in Sierpiński Graphs," *Taiwanese Journal of Matematics,* vol. 9, no. 4, pp. 671–681, 2009.
[8] J. Griggs, and R. Yeh, "Labeling graphs with a Condition at Distance two," *SIAM J. Disc. Math,* vol. 5, no. 4, pp. 586–595, 1992.
[9] W. K. Hale, "Frequency assignment: Theory and Application," *Proc IEEE,* vol. 68, pp. 1479–1514, 1980.
[10] S. Lipschutz and M. Lipson, *Theory and Problems of Discrete Mathematics*, McGraw-Hill, 2007.
[11] A. Lum, "Upper Bound on the L(2,1)-labeling Number of Graphs with Maximum Degree Δ,"
[12] R. Munir, *Matematika Diskrit*, Informatika Bandung, 2003.
[13] S. Paul, M.Pal, and A. Pal, "L(2,1)-labelingg of Circular-arc Graph," *Annals of Pure and Applied Mathematics,* vol. 5, no. 2, pp. 208–219, 2014.
[14] S. G. Rao, *Discrete Mathematical Structures*, New Age International, 2009.
[15] K. H. Rosen, *Discrete Mathematics and Its Applications*, McGraw-Hill, 2012.
[16] Z. Shao, R. Yeh, and D. Zhang, "The L(2,1)-labeling on graphs and the frequency assignment problem," *Elsevier,* vol. 21, pp. 37–41, 2008.
[17] J. J. Siang, *Matematika Diskrit dan Aplikasinya pada Ilmu Komputer*, Andi, 2006.
[18] C. Vasudev, *Graph Theory with Applications,* New Age International, 2006.
[19] W. D. Wallis, *A Beginner's Guide to Graph Theory*, Birkhäuser, 2006.