DESIGNINGSTUDENT WORKSHEET FOR PROBLEM SOLVING TASK THAT ENHANCE STUDENT'S REASONING ABILITY AND CONCEPTUAL UNDERSTANDING

Muliawan Firdaus

Department of Mathematics, State University of Medan
E-mail : feerdhouzt@unimed.ac.id

Abstract

In this paper we detail our approach to designing a student worksheet to be incorporated into problem solving learning scenarios within our ongoing research project. The theoretical influences that inform our approach to design indicate that the processes of creating problem solving task requires a supporting document that would act as a cognitive scaffold for students in the initial stages of the problem solving process before they can internalize the metacognitive strategies and automate the use of these strategies when faced with a new problem. Accordingly, in this paper, we turn our attention to the design of the student worksheetthat can be used as a cognitive scaffold in problem solving tasks. To guide the development of the student worksheet for problem solvingtasks that enhance student's reasoning ability and conceptual understanding we use Pólya's problem solving model and a set of criteria for a good problem develoved by Lappan and Phillips.

Keyword : Student Worksheet, Problem Solving, Reasoning, Conceptual Understanding.

A. Introduction

This paper addresses some of the most central mathematics learning goals: problem solving ability, reasoning ability and conceptual understanding. Problem solving is defined as "engaging in a task for which the solution method is not known in advance" (NCTM, 2000) and includes identifying, posing, and specifying different kinds of problems and solving them, if appropriate, in different ways (Niss, 2003). Reasoning is a fundamental aspect of mathematics (NCTM, 2000). It goes beyond constructing reasoning, and includes abilities like following and assessing chains of arguments, knowing what a proof is and how it differs from other kinds of reasoning, uncovering the basic ideas in a given line of argument, and devising formal and informal arguments (Niss, 2003). The notion of understanding is very complex (Sierpinska 1996), and will be used in a relatively intuitive way, referring to insights in the origin, motivation, meaning and use (Brousseau, 1997) of a mathematical fact, method, concept or other idea.

Mathematical reasoning constitutes a powerful personal and social tool. Today's society, characterized by high demands, competition and constant change related to new scientific and technological developments, requires individuals who, in addition to knowledge, have the ability to solve the challenging problems they face in their lives. Many of those challenges are based in mathematics. It is the role of educators to instill mathematical reasoning abilities, thereby providing students with mathematical tools and processes required to solve everyday problems at home, during leisure and in various fields of work.

The development of a student's mathematical reasoning depends mainly on the experiences that they encounter in life. Students do not develop all of their mathematical abilities by simply memorizing concepts and carrying out routine procedures. Whatever topics are taught, students need to learn them in a way that deepens their mathematical reasoning. An important question for educators is how teachers stimulate students to become thoughtful problem solvers.

Learning mathematical facts and contents is important but is not enough. Students should learn how to use these facts to develop their thinking skills and solve problems. If well facilitated, mathematical problem solving may help students develop and improve the generic ability to solve real life problems (Reys, et.al. 2001), develop critical thinking skills and reasoning (Schafersman, 1991) and gain deep understanding of concepts (Schoen & Charles, 2003).

A fundamental aspect of doing mathematics is to solve mathematics problems, and for this reason, mathematics education was held with the aim that students not only tounderstand mathematics but also to become efficient problem solvers. But in fact, result from preliminary study in our ongoing research we have identified several issues in student's reasoning ability and conceptual understanding that need to be addressed to facilitate the implementation of problem solving in schools.

To aid in the implementation of problem solving in schools, werealize that the design of specific problems or problem solvingtasks cannot be the only focus of problem solving; rather, cognitive scaffolds that allow students to solve a wider range of problems shouldalso be an important focus (Holton & Clarke, 2007). Hannafin et al. (2001) suggest that scaffolding is a process where learners are supportedwhile engaged in a learning or performance task. Traditionally teachers have scaffolded learners todevelop enhanced cognitive structures that assist them to solve problems. By building on theleamer's experiences, providing challenging authentic activities requiring reflective thinking andworking in collaborative groups, teachers can provide the scaffolding needed to bridge the 'zoneofproximal development' (Vygotsky, 1978). Scaffolding is generally regarded as support forlearners while they are engaged in activities just beyond their capabilities. It ranges from assistingwith an entire task to providing occasional support. As the learners' capabilities improve, theteacher gradually reduces the support until the learner becomes self-sufficient with the assignedproblem.Accordingly, in this paper, we turn our attention to the design of the student worksheetthat can be used as a cognitive scaffold in problem solving tasks.

B. Problem Solving Task

In this paper, the term problem solving task refers to mathematical tasks that have the potential to provide intellectual challenges that can enhance students' mathematical development. Such tasks (problems) can promote students' conceptual understanding, foster their ability to reason and communicate mathematically, and capture their interests and curiosity (Hiebert & Wearne, 1993; Marcus & Fey, 2003; NCTM, 1991; van de Walle, 2003). Research recommends that students should be exposed to truly problematic tasks so that mathematical sense making is practiced (Marcus & Fey, 2003; NCTM, 1991; van de Walle, 2003). Mathematical problems that are truly problematic and involve significant mathematics have the potential to provide the intellectual contexts for students' mathematical development. Mathematical problems should be intriguing and contain a level of challenge that invites speculation and hard work so that give students the chance to solidify and extend what they know and stimulate mathematics learning. Most important, mathematical tasks should direct students to investigate important mathematical ideas and ways of thinking toward the learning goals (NCTM, 1991).

To develop mathematical problems that foster students' reasoning ability and conceptual understanding, we adopta set of criteria for a good problem develoved by Lappan and Phillips (1998). The following is such problem criteria:

1. The problem has important, useful mathematics embedded in it.
2. The problem requires higher-level thinking and problem solving.
3. The problem contributes to the conceptual development of students.

4. The problem creates an opportunity for the teacher to assess what his or her students are learning and where they are experiencing difficulty.
5. The problem can be approached by students in multiple ways using different solution strategies.
6. The problem has various solutions or allows different decisions or positions to be taken and defended.
7. The problem encourages student engagement and discourse.
8. The problem connects to other important mathematical ideas.
9. The problem promotes the skillful use of mathematics.
10. The problem provides an opportunity to practice important skills.

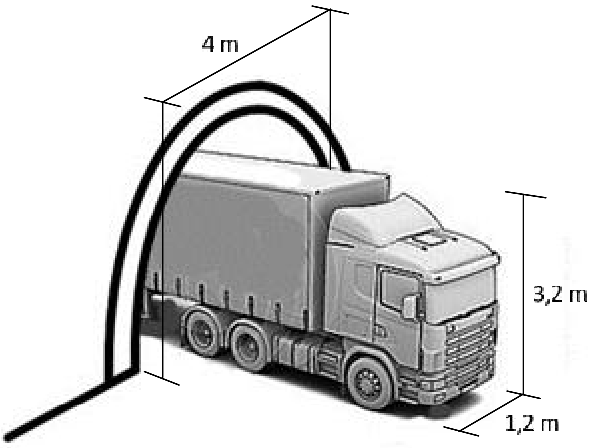
Every problem that a teacher chooses should not have to satisfy all the above criteria; which criteria to consider should depend on a teacher's instructional goals. For example, some problems are used primarily because they provide students with an opportunity to practice a certain skill (criterion 10), whereas others are used primarily to encourage students to collaborate with one another and justify their thinking (criteria 6 and 7). But the first four criteria (important mathematics, higher-level thinking, conceptual development, and opportunity to assess learning) should be considered essential in the selection of all problems. The real value of these criteria is that they provide teachers with guidelines for making decisions about how to make problem solving a central aspect of their instruction.

The role of teachers is to revise, select, and develop tasks that are likely to foster the development of understandings and mastery of procedures in a way that also promotes the development of abilities to solve problems, reason, and communicate mathematically (NCTM, 1991). The following example illustrates how we modified a standard textbook problem in a way that both engages students in learning important mathematics (criterion 1) and also enhances the development of their problem-solving abilities (criteria 2, 3, 4, and 5).

Find the equation of the ellipse which has a minor axis of length 4 *and a vertex at* (0,4).

This kind of problem might be found in any standard textbook. It involves important mathematics, but in its present form, criteria 2, 3, 4, and 5 are not included.By making a revision, teacher can make the open-ended version using of the problem and by doing so raise the cognitive demand (criterion 2) and also satisfy criteria 3 and 4:

A semi-elliptical tunnel whose base has a width of 4 *meters has to be of such a height that a truck with a height of* 3,2 *meters and width of* 1,2 *meters will just fit through it. What is the height of the tunnel*?



This revised example illustrateshow equations, lines, and curves are models of the relationship between two real world quantities; how algebraic procedures and geometric concepts are related; how position in the plane can be represented using rectangular coordinates.We used the open-ended version in revising the original problem becausethe use of open-ended mathematics problems enabled students to develop and stretch their conceptual understanding (Capraro, Cifarelli, Capraro, & Zientek, 2006; Cifarelli & Cai, 2005). Modifying problems that already exist in textbooks is often a relatively easy thing to do but increases the learning opportunity for students.

Open-ended problem solving provides a free and supportive learning environment for students to develop and express their mathematical understandings The educational benefits for students are many. Since open-ended problems allow for different correct solutions, each student has opportunities to obtain her/his own unique solutions. Every student can respond to the problem in some significant way. It is important for every student to be involved in classroom activities and lessons should be understandable for every student. Students have more opportunities to make comprehensive use of their mathematical knowledge and skills. With many different solutions, students can choose their favorite strategies to obtain answers and create their unique solutions. Teachers in turn are able to conduct rich discussions with students that involve the various strategies students used to solve problems. Through comparing and discussing, students are motivated to give other students reasons for their solutions. This affords great opportunities for students to develop their mathematical thinking. Rich experiences allow students to have the pleasure of discovery and receive approval from fellow students (Sawada, 1997).

According to Polya (1945): *"One of the first and foremost duties of the teacher is not to give his students the impression that mathematical problems have little connection with each other, and no connection at all with anything else....The teacher should encourage the students to imagine cases in which they could utilize again the procedure used, or apply the result obtained"*,students can learn to become better problem solvers. Polya’s (1945) presented four phases or areas of problem-solving, which have become the framework often recommended for teaching and assessing problem-solving skills. The four steps are: (1) understanding the problem, (2) devising a plan to solve the problem, (3) implementing the plan, and (4) reflectingon the problem. In our ongoing research, we use Pólya’s problem solving model to guide the development of the problem solvingtasks.

C. Student Worksheet for Problem Solving Task

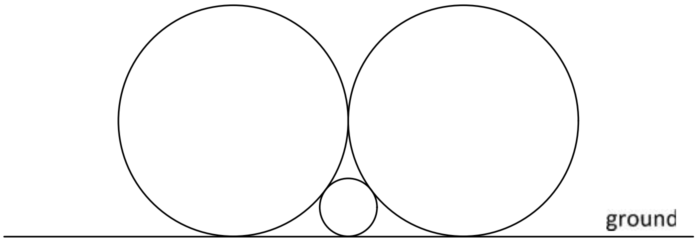
Based on the literature we drafted the following steps in designing a student worksheet for problem solving tasks:

1. Start with the instructional goal.
2. Brainstorm the skills and knowledge needed.
3. Decide how to review or teach the skills.
4. Set up an example.
5. Write the solving steps.
6. Organize questions.

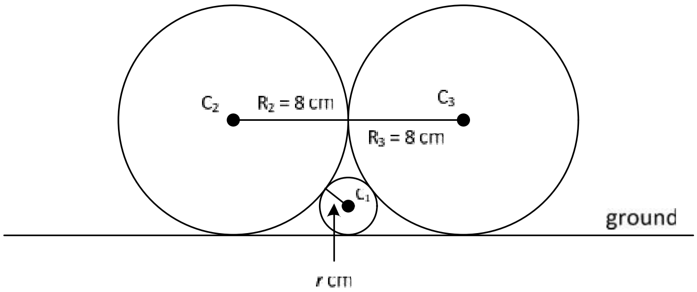
Brainstorming the skills and knowledge needed is useful to complete the instructional goal so teacher can see what he or she might need to review or teach the students in order for them to successfully do the worksheet.In order to prepare students for success with the worksheet, teacher should decide how to review or teach these skills.

A worksheet should includes at least one example and a series of questions to practice the instructional goal. The example is important because it models one way to solve the problem systematically and logically.To set up an example, we includethe question, break down the process of solving the example problem into steps, and explain each step briefly.The following problem and its alternativesolution illustrates how teacher can set up an example.

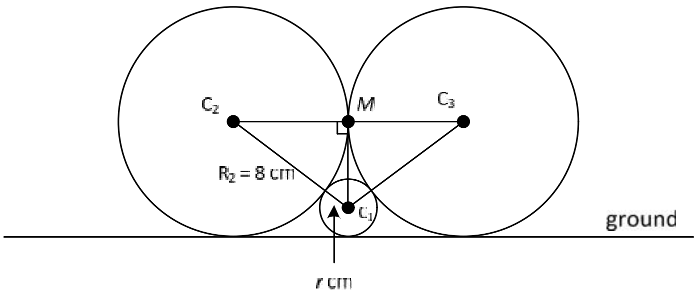
Problem:
Two logs sit side by side so that they are tangent to the ground. Obviously there is enough room between the two logs to place another small log, also tangent to the ground. If the two larger logs are eight centimeters in radius, what is the radius of the smaller log? Generalize so that if the radius is x for the two larger logs, what is the radius of the smaller log?



Solution:
Step1: *understanding the problem*
Logs are the real world models of circles. You can label the figure by first letting C_1 be the smaller circle with r be the radius, C_2 and C_3 be the two larger circles withR_2 and R_3 be the radii of circlesC_2 and C_3 respectively where$R_2 = R_3 = 8$.



Step2: *devising a plan to solve the problem*
You can draw three lines joining each center of the three circle to have an isosceles triangle with vertices C_1, C_2, and C_3.
Let $\overline{C_2C_3}$ be the base of the triagle and M is the midpoint of this base. Then asegment from the center of the smaller circle C_1 to the midpoint M would be the altitude because the triangle $C_1C_2C_3$ is an isosceles triangle. For this reason, you can say that this altitude is prependicular to $\overline{C_2C_3}$. If you draw this altitude, you will have a right triagle C_1MC_2. It is clear that $\overline{C_1M} = R - r$. It is also clear that $\overline{C_1C_2} = r + R_2 = r + R$.



You can apply Pythagoras theorem to triangleC_1MC_2 to find the lengt of $\overline{MC_2}$ and then get the lenght of r.

$$(r+R)^2 = R^2 + (R - r)^2$$

Step 3: *implementing the plan*

$$\begin{aligned} (r+R)^2 &= R^2 + (R - r)^2 \\ (r+8)^2 &= 8^2 + (8 - r)^2 \\ r^2 + 16r + 64 &= 64 + 64 - 16r + r^2 \\ 32r &= 64 \\ r &= 2 \end{aligned}$$

Radius of smaller log is 2 cm.
In general, if the radius is x for the two larger logs, then

$$\begin{aligned} (r+x)^2 &= x^2 + (x - r)^2 \\ r^2 + 2rx + x^2 &= x^2 + x^2 - 2rx + r^2 \\ 4rR &= x^2 \\ r &= \frac{1}{4}x \end{aligned}$$

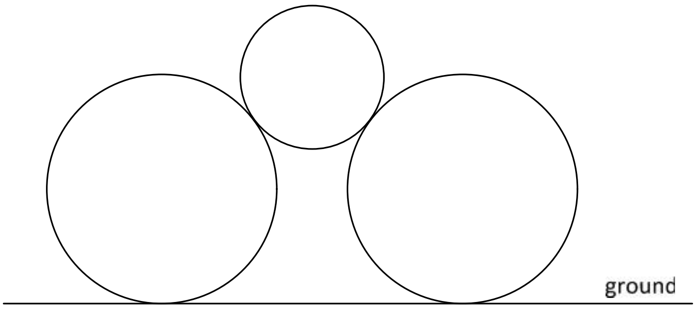
the radius of the smaller log is $\frac{1}{4}$ the radius of the larger logs.

Step 4: *reflectingon the problem*
If the radius of the larger log is 8 then$\frac{1}{4} \times 8 = 2$ is the radius of the smaller logs.

In writing solution steps for student worksheet, theacher should always use the same words to describe a particular action. This consistent wording could increases the student's efficiency in learning, makes patterns in problem solving easier to see, and makes problem solving strategies easier to remember.By breaking a problem down into its steps, we are modeling a strategy that student can apply tosimilar problems. Step-by-step thinking helps student to pay attention to the process that led themto getting the answer. By repeating the same basic steps, a habit is formed and student develop astrategy.

In organizing questions, the first few questions of the worksheet should be the same as the example. This means the wording of thequestions is the same but the numbers are different. Keeping the first few questions the same reinforcesthe skill, provides practice and builds confidence. Teacher can increase the difficulty of the questions by wording the questions differently, increasing the complexity of the numbers, or presenting a more complex situation/diagram/problem.The following is an example of such question.

Two logs have equal radii and are tangent to the ground. Another small log with four centimeters in radius, tangent to the two larger logs. If the two larger logs are eight centimeters in radius and the distance between the centers of themis 18 cm, what is the distance from the center of the smaller log to the ground?Generalize so that if x is the distancebetween the centers of the two larger logs, what is the distance hfrom the center of the smaller log to the ground?



Traditionally, the assessment of problem solving in the classroom has focused on assessing the products rather than the processes of problem solving. The assessment strategy must match it so as to drive the mode of teaching and learning of mathematics.Thus, the process of solving the problem is as important, if not more important, than the final solution of the problem. As such, assessment of problem solving should consider carefully the problem solving process.

D. Conclusion

The development of a student's mathematical reasoning and conceptual understanding depends on the experiences that they encounter in life. Students do not develop all of their mathematical abilities by simply memorizing concepts and carrying out routine procedures. Whatever topics are taught, students need to learn them in a way that deepens their mathematical reasoning. An important question for educators is how teachers stimulate students to become thoughtful problem solvers. The student worksheet holds promise for teachers who want to elevate problem solving to a prominent position in the mathematics classroom. Using the worksheet, teacher can encourage problem solving in their classes. Our task design, which includes the student worksheethas shown great potential in developing student self-scaffolding in problem solving.

E. References

Brousseau, G. (1997). *Theory of didactical situations in mathematics*. Dordrecht: Kluwer AcademicPublishers.

Hannafin, M.J., Hannafin, K. M., McCarthy, J.E., & Radtke, P. (2001). Scaffolding Performance inEPSSs: Bridging Theory and Practice.In Annual Proceedings of the *World Conference onEducational Multimedia, Hypermedia* &*Telecommunications. 658-661.*

Hiebert, J., & Wearne, D. (1993).Instructional task, classroom discourse, and students' learning in second grade. *American Educational Research Journal, 30,* 393-425.

Holton, D., & Clarke, D. (2007). Scaffolding and metacognition. *International Journal of MathemaicsEducation in Science and Technology*, *37*(2), 127-143.

Lappan, G., & Phillips, E. (1998). Teaching and learning in the Connected Mathematics Project. In L. Leutzinger (Ed.), *Mathematics in the middle* (pp. 83-92). Reston, Va.: National Council of Teachers of Mathematics.

Marcus, R., & Fey, J. T. (2003). Selecting quality tasks for problem-based teaching. In H. L. Schoen & R. I. Charles (Eds.), *Teaching mathematics through problem solving: Grades 6-12*(pp. 55-67). Reston, VA: National Council of Teachers of Mathematics.

National Research Council (2001). *Adding It Up: Helping Children Learn Mathematics.*Washington, DC: National Academy Press

NCTM.(1991). *Professional standards for teaching mathematics.* Reston, VA: Author.

NCTM. (2000). *Principles and Standards for School Mathematics*. Reston, Va.: National Council ofTeachers of Mathematics.

Niss, M. (2003). Mathematical competencies and the learning of mathematics: The Danish KOMproject. *Third Mediterranean Conference on Mathematics Education*, Athens (pp. 115–124).

Pólya, G. (1945). *How to solve it: A new aspect of mathematical method.* Princeton, NJ: PrincetonUniversity Press.

Reys, R. E., Lindquist, M. M.,Lambdin, D. V., Smith, N. L. &Suydam, M. N. (2001). *Helping Children Learn Mathematics*6th ed., John Wiley & Sons, Inc., New York.

Schafersman, S. D.(1991). *An Introduction to Critical Thinking*.http://facultycenter.ischool.syr.edu/wp-content/uploads/2012/02/Critical-Thinking.pdf

Schoen, H. L. and Charles, R. I. (2003).*Teaching Mathematics Through Problem Solving*.NCTM.

Sierpinska, A. (1996). *Understanding in mathematics*. Routledge: Falmer.

Van de Walle, J. A. (2003). Designing and selecting problem-based tasks. In F. K. Lester, Jr., & R. I. Charles (Eds.), *Teaching mathematics through problem solving: Prekindergarten-grade 6*(pp. 67-80). Reston, VA: National Council of Teachers of Mathematics.

Vygotsky, L. S. (1978). Interaction between learning and development (M. Lopez-Morillas, Trans.).In M. Cole, V. John-Steiner, S. Scribner, & E. Souberman (Eds.), *Mind in society: Thedevelopment of higher psychological processes* (pp. 79-91). Cambridge, MA: Harvard UniversityPress.